Approximations and definitions of e

This is intended as commentary on section 5.4 of "Irresistible Integrals" by Boros and Moll. They define e by

$$\int_1^e \frac{dt}{t} = 1 \tag{1}$$

Then, their proposition 5.4.1 is that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \tag{2}$$

In other words, the idea is to connect two familiar aspects of the number e. I find their proof was not intuitive, so here is a different approach. We might as well define the natural logarithm:

$$\ln x \equiv \int_1^x \frac{dt}{t} \tag{3}$$

Then

$$\ln \left(1 + \frac{1}{n}\right)^n = n \ln \left(1 + \frac{1}{n}\right) = \frac{\ln\left(1 + \frac{1}{n}\right) - \ln 1}{1/n}, \tag{4}$$

and we see that as n grows, $1/n$ plays the role of an infinitesimal in a difference quotient. Therefore

$$\lim_{n\to\infty} \ln \left(1 + \frac{1}{n}\right)^n = \left.\frac{d}{dx}\right|_{x=1} \ln x = 1 = \ln e \tag{5}$$

By the continuity of the natural logarithm function evaluated at 1 (which is evident from the definition, since it is even differentiable there),

$$\ln \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n\right) = \lim_{n\to\infty} \ln \left(1 + \frac{1}{n}\right)^n = \ln e \tag{6}$$

We now have two quantities whose logarithms are equal. The natural logarithm function over the positive real numbers is obviously injective, since it is a strictly increasing function (being the integral of a positive function). Therefore we have established that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \tag{7}$$

We used the fact that $\ln x^n = n \cdot \ln x$ as well as the fact that the derivative of $\ln x$ is $1/x$. Both of these follow straightforwardly from our integral definition of $\ln x$ without assuming what we wanted to prove. In particular, using the change of variable $u = t^n$ gives

$$\ln x = \int_1^x \frac{dt}{t} = \int_1^{x^n} \frac{du}{nt^{n-1} \cdot t} = \frac{1}{n} \ln x^n \tag{8}$$

Meanwhile, the fact that the derivative of $\ln x$ is $1/x$ follows from the integral definition of the natural logarithm via the fundamental theorem of calculus.

I find this proof of Eq. 7 to be better motivated than the one given by Boros and Moll. However, theirs has the advantage that it also shows that $(1+1/n)^n$ increases with n. I've been trying to find a different way to show this, which has led to some interesting things. For example, a first attempt would be to take the derivative with respect to n and see if we can show it to be positive. Or, more simply, show that the derivative of the logarithm is positive:

$$\begin{aligned} \frac{d}{dn} n \ln\left(1+\frac{1}{n}\right) &= \ln\left(1+\frac{1}{n}\right) + n\frac{n}{n+1}\cdot\left(\frac{-1}{n^2}\right) \\ &= \ln\left(1+\frac{1}{n}\right) - \frac{1}{n+1} \end{aligned} \tag{9}$$

If you set x equal to $1/n$ you see that what we are trying to show is equivalent to

$$\frac{x}{x+1} < \ln(1+x) \tag{10}$$

Let's keep that in mind as something to prove later. Meanwhile, in Boros and Moll's Problem 5.4.1 part b, we're asked to prove that while $(1+1/n)^n$ increases with n (reaching e from below), $(1+1/n)^{n+1}$ decreases with n (reaching e from above). If we try to apply the same derivative trick as before, we arrive at

$$\frac{d}{dn}(n+1)\ln\left(1+\frac{1}{n}\right) = \ln\left(1+\frac{1}{n}\right) - \frac{1}{n} \tag{11}$$

Once again setting $x = 1/n$, we see that proving this derivative to be negative is equivalent to proving the inequality

$$\ln(1+x) < x \tag{12}$$

So sum up this last part, we have seen that showing the sequence $(1+1/n)^n$ to be monotonically increasing and the sequence $(1+1/n)^{n+1}$ to be monotonically decreasing is equaivalent to establishing the bounds

$$\frac{x}{x+1} < \ln(1+x) < x \tag{13}$$

These seem tractable, perhaps by showing that certain second derivatives are strictly positive. Meanwhile, however, let's look at another attempt to show the monotonicity of those two sequences. If $(1+1/n)^n$ increases with n, that means, equivalently, that:

$$1 < \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} = \left[\frac{n(n+2)}{(n+1)^2}\right]^n \left(\frac{n+2}{n+1}\right) \tag{14}$$

It is interesting to note that the quantity in square brackets is less than 1 by the inequality between geometric and arithmetic means. And that quantity is raised to the power n, meaning that it is the factor $(n+2)/(n+1)$ that saves this expression from being less than 1. Proving the inequality above would give us the following strange "anti AM-GM inequality":

$$\sqrt{n(n+2)} > \left(\frac{n+1}{n+2}\right)^{1/2n} (n+1), \tag{15}$$

which sets a limit on how far the geometric mean of n and $n+2$ can lag behind the arithmetic mean $(n+1)$. I wonder if this can be generalized to all pairs of real numbers.

And what of the fact that $(1+1/n)^{n+1}$ is monotonically decreasing? It is equivalent to

$$1 > \frac{(1+\frac{1}{n+1})^{n+2}}{(1+\frac{1}{n})^{n+1}} = \left[\frac{n(n+2)}{(n+1)^2}\right]^{n+1} \left(\frac{n+2}{n+1}\right) \tag{16}$$

This is the same as before, but with an extra power of the quantity in square bracksets, which is enough to reverse the sign of the inequality. We can put these two inequalities together to get

$$1 < \left[\frac{(n+1)^2}{n(n+2)}\right]^n < \left(\frac{n+2}{n+1}\right) < \left[\frac{(n+1)^2}{n(n+2)}\right]^{n+1} \tag{17}$$

I don't know whether this is neat or whether it will seem obvious once I stare at it long enough. In any case we still haven't proven it here! Let's return to the equivalent inequalities:

$$\frac{x}{x+1} < \ln(1+x) < x \tag{18}$$

We are particularly interested in establishing these for $x > 0$ since for us $x = 1/n$ where n is a positive integer. It seems straightforward now: define the functions

$$f(x) = x - \ln(1+x) \tag{19}$$

$$g(x) = \ln(1+x) - \frac{x}{x+1} \tag{20}$$

Note that $f(0) = g(0) = 0$ and that for $x > 0$,

$$f'(x) = 1 - \frac{1}{1+x} > 0 \tag{21}$$

$$g'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} > 0 \tag{22}$$

It follows that these functions are positive for $x > 0$. And our other strange inequalities follow as well.

Another question: if $(1+1/n)^n$ is monotonically increasing and $(1+1/n)^{n+1}$ is monotonically decreasing, what about the sequences $(1+1/n)^{n+\alpha}$ for $0 < \alpha <$

1? Are they non-monotonic for certain values of α or do they switch from monotonically increasing to monotonically decreasing for some threshold value of α? If so, does that give rise to a much better (faster converging) approximation to e than the sequences with $\alpha = 0$ or 1? From some numerical experimentation, it seems that $\alpha = 1/2$ might produce a faster converging series. In fact, here is a way to see this: for a given value of n, pick the α that makes the quantity $(1 + 1/n)^{n+\alpha}$ exactly equal to e, which is to say that

$$(n + \alpha_n)\ln(1 + \frac{1}{n}) = 1 \tag{23}$$

$$\alpha_n = \frac{1}{\ln(1 + \frac{1}{n})} - n \tag{24}$$

This value α_n gives the best possible approximation to e for a given n. In fact, it gives exactly e. So what happens when we let n go to infinity? Equivalently, let's once again let $x = 1/n$ and ask for the limit

$$\lim_{x\to 0}\left[\frac{1}{\ln(1+x)} - \frac{1}{x}\right] = \lim_{x\to 0}\left[\frac{x - \ln(1+x)}{x\ln(1+x)}\right] \tag{25}$$

The numerator and denominator both vanish at $x = 0$. So do their derivatives! Only after applying l'Hôpital's rule twice do we finally find that

$$\lim_{n\to\infty}\alpha_n = \frac{1}{2} \tag{26}$$

So I would guess that a better approximation to e might be given by

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n+1/2} \tag{27}$$

Indeed, Mathematica tells me that

$$\lim_{n\to\infty}\frac{\ln\left[e - \left(1+\frac{1}{n}\right)^{n}\right]}{\ln n} = -1 \tag{28}$$

$$\lim_{n\to\infty}\frac{\ln\left[e - \left(1+\frac{1}{n}\right)^{n+1/2}\right]}{\ln n} = -2 \tag{29}$$

It also seems to show that only with $\alpha = 2$ do the errors decrease as $1/n^2$ rather than $1/n$. But how to prove this?

I have come across a Stack Exchange question (called "Proving $(1 + 1/n)^{n+1} > e$)" where someone more or less answers this question. Rather than looking at the complicated limits above, let's just look at the Taylor expansion of

$$\begin{aligned}(1 + \frac{1}{n})^{n+\alpha} &= \exp\left[(n+\alpha)\ln(1 + \frac{1}{n})\right] \\ &= \exp\left[(n+\alpha)(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots)\right] \\ &= \exp\left[1 + (\alpha - \frac{1}{2})\frac{1}{n} + (\frac{1}{3} - \frac{\alpha}{2})\frac{1}{n^2} + \ldots\right]\end{aligned} \tag{30}$$

This makes it plain that the leading term in the difference between $(1 + 1/n)^{n+\alpha}$ and e is of order $1/n$ if $\alpha \neq 1/2$ and of order $1/n^2$ in the special case $\alpha = 1/2$.